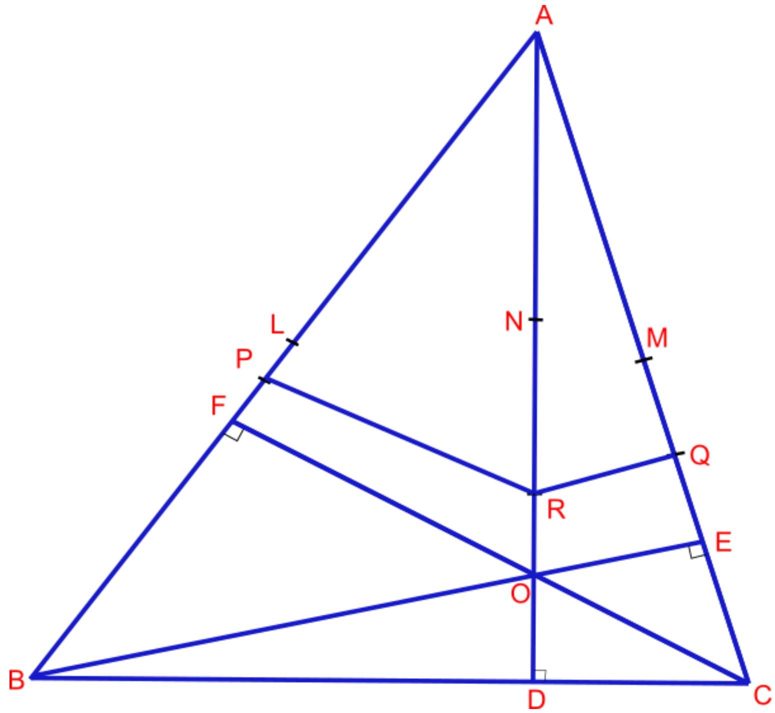
Cash Award Question of 01.09.2023



In the picture, AD, BE & CF are the altitudes of ΔABC and O is its Orthocentre. L, M & N are the midpoints of AB,AC & AO respectively. P, Q & R are the midpoints of LF, ME & ND respectively. Prove: APRQ is concyclic.

Question framed by
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Author's Solution

Before going to the solution, let us see an important theorem, which will be later used in the solution. (This theorem was already proved vide a problem given in one of the previous month's contest)

CONCURRENT CHORDS THEOREM:

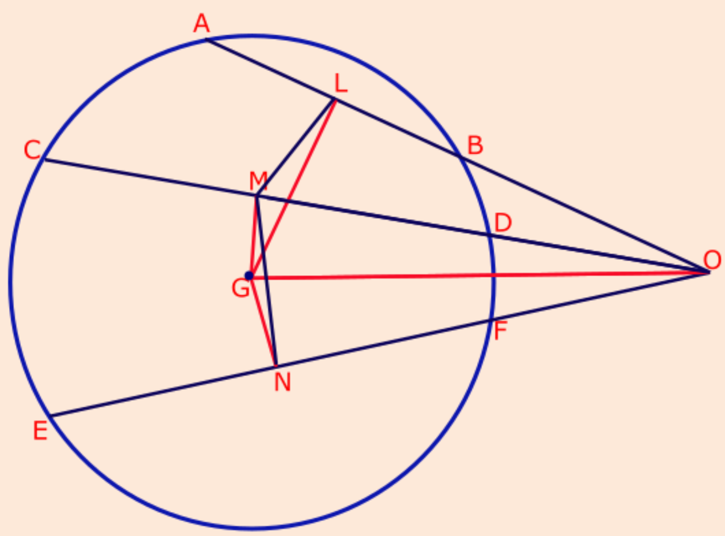
If three or more chords are concurrent at a point inside or outside their circle, then the midpoints of those chords and the said point of concurrency are concyclic.

Given: (In the picture)

AB, CD & EF are chords concurrent at O. [in this case 'O' is outside the circle]

L, M & N are midpoints of AB, CD & EF respectively.

To Prove: OLMN is concyclic



Construction : **Let 'G' be the centre of the circle. Join GL, GM & GN.**

Proof:

GL ⊥ AB , GM ⊥ CD & $GN \perp EF$ [∵L, M & N are midpoints of the chords and G is the circumcentre]

$\Longrightarrow \angle OLG + \angle ONG = 90° + 90° = 180°$

∴ $OLGN$ is concyclic -------------------------------- (1)

$\Longrightarrow \angle OMG + \angle ONG = 90° + 90° = 180°$

∴ OMGN is concyclic ----------------------------------(2)

In (1) & (2), three points viz O,G & N are common

∴ OLMGN are concyclic ------------------------Proved

Corollary :

OG is the diameter of the concyclic circle of OLMGN

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Solution:

To Prove : APRQ is concyclic

Construction :

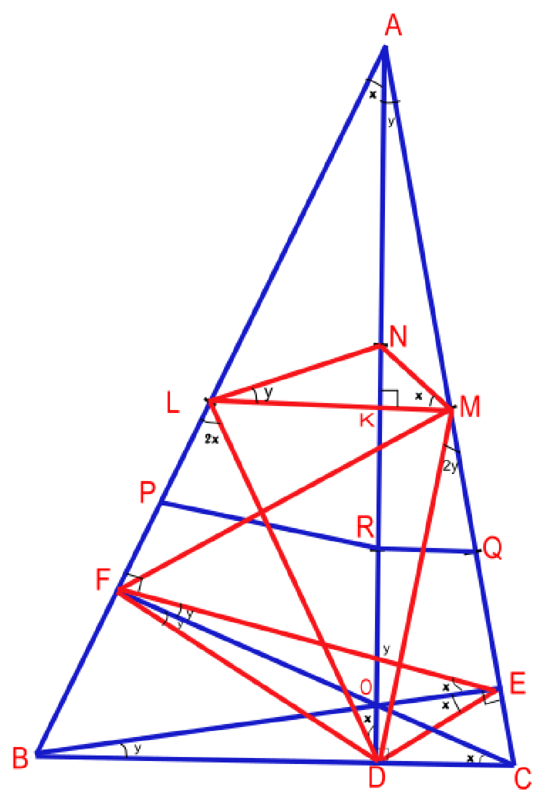
Join DE, EF, FD, LD, DM, FM, LN & MN.

Let LM & AD intersect at K.

Proof:

$\Delta\, DEF$ is the orthic triangle of ΔABC.

∴ DO,EO & FO are the angle bisectors of its angles at D, E & F.



Let $\angle BAD = x$ & $\angle EAD = y$

AFDC is concylic (∵ $\angle AFC = \angle ADC = 90°$)

$\Longrightarrow \angle FAD = \angle FCD = x$

Similarly, BFEC is concyclic

$\Longrightarrow \angle FEB = \angle FCB = x$

$\Longrightarrow \angle FED = 2x$

L & M are midpoints of AB & AC

$\therefore LM \parallel BC$

And LK is the perpendicular bisector of AD.

$\therefore \angle BLD = \angle LAD + \angle LDA = 2x = \angle FED$

$\Longrightarrow$ $LFDE$ **is concyclic -----------------------------(1)**

N is midpoint of AO (Given)

$\therefore$ **LN**$\parallel BO$, **MN**$\parallel CO$ **& BC** $\parallel LM$

$\therefore \Delta LNM \sim \Delta BOC$

$\Longrightarrow \angle LMN = \angle BCO = x = \angle LDN$

$\Longrightarrow$ $LDMN$ **is concyclic ----------------------------------(2)**

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Similarly, $\angle EMD = \angle EFD = 2y$

$\Longrightarrow$ $FDEM$ **is concyclic ------------------------- (3)**

(1) & (3) ⟶ LFDEM is concyclic ------------- (4) (3 points common in both)

(4) & (2) ⟶LFDEMN is concyclic -------------(5) (3 points common in both)

(5) ⟶

FL, DN & EM are chords of the same circle concurrent at A (outside the circle)

$\therefore$ **Their midpoints P, Q & R and their point of concurrency A are concyclic [By concurrent Chords Theorem Proved above]**

ie.. APRQ is concyclic.

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